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Uniqueness in Linear Semi-infinite Optimization

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This paper is concerned with the problem of uniqueness in linear semi-infinite optimization. General characterization theorems are given for problems with continuous and differentiable functions. The relationship between linear semi-infinite optimization and one-sided L_1-approximation is also used in order to derive certain characterizations. © 1993 Academic Press, Inc.

INTRODUCTION

This paper is concerned with linear semi-infinite optimization. Let us first introduce the problem.

Let K be a compact subset of $\mathbb{R}^d$, $d \geqslant 1$, such that $\overline{\text{int}(K)} = K$ and let us denote by $C(K)$ the set of continuous functions defined on K. Suppose that u_i^j are functions in $C(K)$ for $i = 1, ..., n$ and $j = 1, ..., m$. Moreover, let $f_j \in C(K)$, $j = 1, ..., m$, and $\mathbf{p} = (p_1, ..., p_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. Then we consider the following problem.

PROBLEM I. *Minimize* $p(\mathbf{a}) = \sum_{i=1}^n a_i p_i$, $\mathbf{a} = (a_1, ..., a_n)$, *in* $\mathbb{R}^n$, *subject to the constraints*

$$\sum_{i=1}^{n} a_i u_i^j(x) \leqslant f_j(x), \qquad x \in K, j = 1, ..., m.$$

This is a standard problem in linear semi-infinite optimization (e.g., see [1, 2]). Many results are given concerning existence and characterization of this problem.

Let $\mathbf{p}$, $\{u_i^j\}$, and $\{f_j\}$ be arbitrarily given but fixed in this problem. Then characterizations of uniqueness are known. However, a solution $\mathbf{a}^*$ of the

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problem is used in these characterizations. In this paper we shall be concerned with developing a general theory of uniqueness in linear semi-infinite optimization. In this sense our paper is closer in spirit to the Haar theory in linear Chebyshev approximation. Our goal is to provide conditions ensuring uniqueness of the solutions of Problem I which only use properties of the functions $\{u_i^j\}$ and are independent of the functions $\{f_j\}$.

This problem has only been considered for the case in which $m = 1$ and the functions in the problem are continuous. In this case there is no uniqueness in general (see [7]). But the situation is radically altered if $m > 1$ or if we restrict ourselves in special cases to differentiable functions.

We give a complete characterization of uniqueness of the solutions in Problem I for given $\mathbf{p}$ and $\{u_i^j\}$ and all choices of functions $\{f_j\}$ satisfying the so-called Slater condition. It turns out that requiring uniqueness for all continuous functions is too restrictive in many cases. Therefore we also consider the case of smooth functions where we obtain meaningful results. In the case $m = 1$ we extend our problem as follows. Suppose that $\mathbf{P}$ is the set of vectors $\mathbf{p}$ for which the solution set of Problem I is nonempty and bounded for given $\{u_i^j\}$ and some functions $\{f_j\}$. We develop conditions ensuring uniqueness of solutions in Problem I for all $\mathbf{p} \in \mathbf{P}$ and all functions f_1; i.e., only the functions $\{u_i^1\}$ are fixed. These investigations have been inspired by similar considerations in one-sided L_1-approximation. It is well known that there is a close relationship between one-sided L_1-approximation and linear semi-infinite optimization. We use many results which have recently been given in one-sided L_1-approximation (for a survey see [8]) in order to obtain uniqueness results in semi-infinite optimization. In particular, it can be shown that in the case in which $\{u_j^1\}_{j=1}^n$ and f_1 are differentiable functions we obtain many important examples of uniqueness. But we are not able to extend these characterizations to the case $m > 1$. We show by the example of linear Chebyshev approximation with constraints that the situation is very complicated for $m > 1$.

The paper is organized as follows. In Section 1 we first state some results concerning existence and characterization of the solutions of Problem I. Then a general characterization of uniqueness of solutions for all continuous (differentiable) functions is shown. In Section 2 the case $m = 1$ is studied in more detail. We also give estimates for the dimension of the solution set if the solution is not unique.

1. A General Characterization Theorem of Uniqueness

In this section we first state some results concerning existence and characterization of solutions. The following condition is of importance in this context.

Let $\{u_i^j\}_{j=1}^{m}{}_{i=1}^{n}$ and $\{f_j\}_{j=1}^{m}$ in $C(K)$ be arbitrarily given but fixed functions in Problem I. Then the problem is said to satisfy the *Slater condition* if there is a vector $\mathbf{a}=(a_1, ..., a_n)$ in $\mathbb{R}^n$ satisfying

$$\sum_{i=1}^{n} a_i u_i^j(x) < f_j(x), \qquad x \in K,\ j=1, ..., m.$$

We first consider the problem of existence of solutions. For this reason we need the following notation.

Definition 1.1. Let the functions $\{u_1^j, ..., u_n^j\}$, $j=1, ..., m$, in $C(K)$ be given. Then we denote by $\mathbf{P}$ the set of all vectors $\mathbf{p}=(p_1, ..., p_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that if $\sum_{i=1}^{n} b_i u_i^j(x) \leqslant 0$ for all $x \in K$, $j=1, ..., m$ (where $\mathbf{b}=(b_1, ..., b_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$), then $p(\mathbf{b})=\sum_{i=1}^{n} b_i p_i > 0$.

The following is true.

Theorem 1.2. *Let Problem I be given for some* $\mathbf{p} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ *and suppose that there is an* $\mathbf{a}=(a_1, ..., a_n)$ *satisfying* $\sum_{i=1}^{n} a_i u_i^j(x) \leqslant f_j(x)$, $j=1, ..., m$. *Then* $\mathbf{p} \in \mathbf{P}$ *if and only if the solution set of Problem I is nonempty and bounded.*

This theorem follows from a result in [2, p. 71].

In particular, we shall be interested in problems which have a unique solution. Theorem 1.2 shows that we can restrict ourselves to problems satisfying $\mathbf{p} \in \mathbf{P}$.

We shall also state the well-known Kuhn–Tucker theorem characterizing solutions of Problem I (see [1, 2]).

We set for every $f \in C(K)$

$$Z(f)=\{x : f(x)=0\}.$$

Theorem 1.3. *Let Problem I satisfying the Slater condition be given. Then* $\mathbf{a}^*=(a_1^*, ..., a_n^*)$ *is a solution if and only if there exist nonnegative integers* r_j *with* $\sum_{j=1}^{m} r_j \leqslant n$, *points* $\{x_k^j\}_{k=1}^{r_j} \subset Z(f_j - v_j^*)$, $v_j^*=\sum_{i=1}^{n} a_i^* u_i^j$, *and positive numbers* $\{\lambda_k^j\}_{k=1}^{r_j}$ *for all* $j=1, ..., m$, *such that*

$$\sum_{j=1}^{m} \sum_{k=1}^{r_j} \lambda_k^j u_i^j(x_k^j) = -p_i, \qquad i=1, ..., n.$$

The following result concerning uniqueness is well known (see [2, p. 49]):

Let Problem I satisfying the Slater condition be given. Then $\mathbf{a}^*=$

$(a_1^*, ..., a_n^*)$ is a (strongly) unique solution if and only if there does not exist a $\mathbf{b} \in (b_1, ..., b_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ satisfying $p(\mathbf{b}) \leqslant 0$ and

$$\sum_{i=1}^{n} b_i u_i^j(x) \leqslant 0 \qquad \text{for all} \quad x \in Z\left(f_j - \sum_{i=1}^{n} a_i^* u_i^j\right), \; j = 1, ..., m.$$

In this characterization it is necessary to determine the zeros of the functions $f_j - \sum_{i=1}^{n} a_i^* u_i^j$; i.e., the solution $\mathbf{a}^*$ is used in this characterization.

Our main purpose is to give conditions which only use properties of the functions $\{u_i^j\}$ and therefore are independent of the functions $\{f_j\}$.

To this end, we first show the following lemma (see also [8, 9]).

LEMMA 1.4. *Let the functions* $\{u_1^j, ..., u_n^j\}$ *be in* $C(K)$, $j = 1, ..., m$, *and* $\mathbf{p} = (p_1, ..., p_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ *be given. Suppose that the* $(A_j)_{j=1}^m$ *are closed subsets of K such that*

$$\sum_{i=1}^{n} b_i u_i^j(x) \leqslant 0, \qquad x \in A_j, \; j = 1, ..., m$$

implies that

$$p(\mathbf{b}) = \sum_{i=1}^{n} b_i p_i \geqslant 0.$$

Then there exist non-negative integers r_j *with* $\sum_{j=1}^m r_j \leqslant n$, *points* $\{x_k^j\}_{k=1}^{r_j}$ *in* A_j, *and positive numbers* $\{\lambda_k^j\}_{k=1}^{r_j}$ *for all* $j = 1, ..., m$, *such that*

$$\sum_{j=1}^{m} \sum_{k=1}^{r_j} \lambda_k^j u_i^j(x_k^j) = -p_i, \qquad i = 1, ..., n.$$

Proof. Let

$$H = \{(u_1^j(x), ..., u_n^j(x)) : x \in A_j, \quad j = 1, ..., m\}$$

and let G denote the closed convex cone generated by H. By assumption $p \neq \mathbf{0}$. If $-\mathbf{p} \in G$ then the above representation of $-\mathbf{p}$ exists. Therefore we assume that $-\mathbf{p} \notin G$. Since G and $\{-\mathbf{p}\}$ are closed subsets and G is a cone there exists a separating hyperplane through the origin. Hence there is a $\mathbf{c} = (c_1, ..., c_n) \neq 0$ such that

$$\sum_{i=1}^{n} c_i q_i \leqslant 0 < \sum_{i=1}^{n} (-c_i p_i)$$

for all $\mathbf{q} = (q_1, ..., q_n) \in G$. Set $h_j = \sum_{i=1}^{n} c_i u_i^j$, $j = 1, ..., m$. Then $h_j(x) \leqslant 0$ for all $x \in A_j$, $j = 1, ..., m$, and $p(\mathbf{c}) < 0$. This is a contradiction and hence $-\mathbf{p} \in G$. ∎

As we have already mentioned we will obtain unicity of the solutions of Problem I for many classes of differentiable functions. Therefore we give a characterization of uniqueness including also differentiable functions.

In this case we must restrict ourselves to a more special subset K. We assume that K is a compact convex subset of $\mathbb{R}^d$, $d \geqslant 1$, with piecewise smooth boundary. Let $C^1(K)$ be the set of continuously differentiable functions in $C(K)$.

Then let $Z_1(f)$ denote the set of zeros of K for which the following hold:

(1) If $x \in \operatorname{int}(K)$, then all first partial derivatives of f at x vanish (i.e., the gradient of f at x is zero).

(2) If $x \in \partial K$, then all directional derivatives to f at x vanish for all directions tangent to K at x.

THEOREM 1.5. *Let Problem I, $n \geqslant 2$, be given and assume that $\mathbf{p} \in \mathbf{P}$. Let K be a compact convex subset of $\mathbb{R}^d$ with piecewise smooth boundary. Suppose that $\{u_1^j, ..., u_n^j\}$ are in $C(K)$ for $j = 1, ..., m_1$, $0 \leqslant m_1 \leqslant m$, and in $C^1(K)$ for $j = m_1 + 1, ..., m$. Then Problem I has a unique solution for all f_j in $C(K)$, $j = 1, ..., m_1$, and f_j in $C^1(K)$, $j = m_1 + 1, ..., m$, satisfying the Slater condition if and only if for all $\mathbf{a} = (a_1, ..., a_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $v_j = \sum_{i=1}^n a_i u_i^j$, there exists a $\mathbf{b} = (b_1, ..., b_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $w_j = \sum_{i=1}^n b_i u_i^j$, such that*

(a) $w_j(x) \leqslant 0$ *for all* $x \in Z(v_j)$, $j = 1, ..., m_1$;

(b) $w_j(x) \leqslant 0$ *for all* $x \in Z_1(v_j)$, $j = m_1 + 1, ..., m$;

(c) $p(\mathbf{b}) = \sum_{i=1}^n b_i p_i < 0$.

Proof. Assume that (a)–(c) hold and that there exist two solutions $\mathbf{a}^1$ and $\mathbf{a}^2$, $\mathbf{a}^1 \neq \mathbf{a}^2$. Set $\mathbf{a}^* = (\mathbf{a}^1 - \mathbf{a}^2)/2$, $s_j = \sum_{i=1}^n a_i^* u_i^j$, and $h_j = f_j - \sum_{i=1}^n (a_i^1 + a_i^2) u_i^j/2$, $j = 1, ..., m$. Then

$$h_j(x) \geqslant \left| \sum_{i=1}^n a_i^* u_i^j(x) \right|, \qquad x \in K, \quad j = 1, ..., m.$$

We consider Problem I': Minimize

$$p(\mathbf{a}) = \sum_{i=1}^n a_i p_i$$

subject to the constraints

$$\sum_{i=1}^n a_i u_i^j(x) \leqslant h_j(x), \qquad x \in K, \quad j = 1, ..., m.$$

It follows that $\mathbf{0}$ and $\mathbf{a}^*$ are solutions of Problem I'. Since $h_j \in C(K)$, $j = 1, ..., m_1$, and $h_j \in C^1(K)$, $j = m_1 + 1, ..., m$, we obtain that $Z(h_j) \subset Z(s_j)$,

$j=1, \ldots, m_1$, and $Z(h_j)=Z_1(h_j)\subset Z_1(s_j)$, $j=m_1+1, \ldots, m$. By assumption, there exists a $\mathbf{b}\in\mathbb{R}^n\setminus\{\mathbf{0}\}$ such that the functions $w_j=\sum_{i=1}^n b_i u_i^j$, $j=1, \ldots, m$, satisfy

$$\begin{aligned} &w_j(x)\leqslant 0, \qquad x\in Z(s_j), \quad j=1, \ldots, m_1\\ &w_j(x)\leqslant 0, \qquad x\in Z_1(s_j), \quad j=m_1+1, \ldots, m,\\ &p(\mathbf{b})<0. \end{aligned}$$

Since Problem I satisfies the Slater condition there exists an $\tilde{\mathbf{a}}=(\tilde{a}_1, \ldots, \tilde{a}_n)$ such that

$$\sum_{i=1}^n \tilde{a}_i u_i^j(x)<f_j(x), \qquad x\in K, \quad j=1, \ldots, m.$$

Hence $t_j(x)=\sum_{i=1}^n c_i u_i^j(x)$ satisfies $t_j(x)<0$, $x\in Z(h_j)$, $j=1, \ldots, m$, where $c_i=\tilde{a}_i-(a_i^1+a_i^2)/2$. Let $\mathbf{c}=(c_1, \ldots, c_n)$. Therefore we have for a sufficiently small $\varepsilon>0$ that

$$\begin{aligned} &p(\mathbf{b}+\varepsilon\mathbf{c})<0,\\ &w_j(x)+\varepsilon t_j(x)<h_j(x) \qquad \text{for all} \quad x\in Z(h_j), \quad j=1, \ldots, m, \end{aligned}$$

and for sufficiently small $\delta>0$

$$\begin{aligned} &p(\delta(\mathbf{b}+\varepsilon\mathbf{c}))<0\\ &\delta(w_j(x)+\varepsilon t_j(x))\leqslant h_j(x) \qquad \text{for all} \quad x\in K, \quad j=1, \ldots, m. \end{aligned}$$

Therefore $\mathbf{0}$ is not a solution to Problem I', and this is a contradiction.

To prove the converse we assume that (a)–(c) do not hold. Hence there is a vector $\mathbf{a}^*=(a_1^*, \ldots, a_n^*)\in\mathbb{R}^n\setminus\{\mathbf{0}\}$ such that for all $\mathbf{b}=(b_1, \ldots, b_n)\in\mathbb{R}^n\setminus\{\mathbf{0}\}$, where the functions $v_j=\sum_{i=1}^n a_i^* u_i^j$ and $w_j=\sum_{i=1}^n b_i u_i^j$, $j=1, \ldots, m$, satisfy

$$\begin{aligned} &w_j(x)\leqslant 0, \qquad x\in Z(v_j), \quad j=1, \ldots, m_1,\\ &w_j(x)\leqslant 0, \qquad x\in Z_1(v_j), \quad j=m_1+1, \ldots, m, \end{aligned}$$

we have that $p(\mathbf{b})\geqslant 0$.

Then it follows from Lemma 1.4 that there exist points $\{x_k^j\}_{k=1}^{r_j}$ in $Z(v_j)$, $j=1, \ldots, m_1$, and in $Z_1(v_j)$, $j=m_1+1, \ldots, m$, positive numbers $\{\lambda_k^j\}_{k=1}^{r_j}$, $j=1, \ldots, m$, and non-negative r_j with $\sum_{j=1}^m r_j\leqslant n$ satisfying

$$\sum_{j=1}^m \sum_{k=1}^{r_j} \lambda_k^j u_i^j(x_k^j)=-p_i, \qquad i=1, \ldots, n.$$

If we now assume that there exist functions $\{f_j\}_{j=1}^{m_1}$ in $C(K)$ and $\{f_j\}_{j=m_1+1}^{m}$ in $C^1(K)$ such that

$$f_j(x_k^j)=0, \qquad k=1, ..., r_j \quad \text{and} \quad j=1, ..., m$$

$$\left|\sum_{i=1}^{n} a_i^* u_i^j(x)\right| \leqslant f_j(x), \qquad x \in K, \quad j=1, ..., m$$

then $\pm\mathbf{a}^*$ are solutions of Problem I for these functions $\{f_j\}_1^m$. This is a contradiction. It remains to show that functions $\{f_j\}$ of the above form exist. The construction is difficult but it follows in the same way as in [8, 9]. ∎

Remark. If in Theorem 1.5 we have $m=m_1$, i.e., we only consider continuous functions, then it is sufficient to assume that K is a compact set in $\mathbb{R}^d$ satisfying $\overline{\text{int}(K)}=K$.

From the proof of Theorem 1.5 we immediately obtain the following characterization.

COROLLARY 1.6. *Let the assumptions of Theorem 1.5 be given. Then Problem I has a unique solution for all f_j in $C(K)$, $j=1, ..., m_1$, and f_j in $C^1(K)$, $j=m_1+1, ..., m$, satisfying the Slater condition if and only if there do not exist a vector $\mathbf{a}=(a_1, ..., a_n)\in\mathbb{R}^n\setminus\{\mathbf{0}\}$, $v_j=\sum_{i=1}^n a_i u_i^j$, points $\{x_k^j\}_{k=1}^{r_j}\in Z(v_j)$, $j=1, ..., m_1$, $\{x_k^j\}_{k=1}^{r_j}\in Z_1(v_j)$, $j=m_1+1, ..., m$, positive numbers $\{\lambda_k^j\}_{k=1}^{r_j}$, $j=1, ..., m$, and non-negative integers r_j with $\sum_{j=1}^m r_j\leqslant n$ satisfying*

$$\sum_{j=1}^{m}\sum_{k=1}^{r_j} \lambda_k^j u_i^j(x_k^j) = -p_i, \qquad i=1, ..., n.$$

2. UNIQUENESS IN THE CASE $m=1$

In this section we consider the special case of Problem I where $m=1$. This problem is usually considered in linear semi-infinite optimization. We state the problem in this simpler form once again.

PROBLEM II. *Let K be a compact subset of $\mathbb{R}^n$, $d\geqslant 1$, $\overline{\text{int}(K)}=K$, be given. Let f be a function in $C(K)$ and $U=\text{span}\{u_1, ..., u_n\}$, $n\geqslant 2$, be an n-dimensional subspace of $C(K)$ which contains a strictly positive function. Suppose that $\mathbf{p}=(p_1, ..., p_n)\in\mathbb{R}^n\setminus\{\mathbf{0}\}$. Minimize $p(\mathbf{a})=\sum_{i=1}^n a_i p_i$, $\mathbf{a}=(a_1, ..., a_n)$, subject to the constraints*

$$\sum_{i=1}^{n} a_i u_i(x) \leqslant f(x), \qquad x\in K.$$

Note that $\mathbf{P}$ is the set of all vectors $\mathbf{p} = (p_1, ..., p_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $\sum_{i=1}^n b_i u_i(x) \leqslant 0$, $x \in K$, and $\mathbf{b} \neq 0$ implies $p(\mathbf{b}) = \sum_{i=1}^n b_i p_i > 0$. The condition that U contains a strictly positive function is equivalent to the condition that Problem II satisfies the Slater condition with respect to all choices of f.

Let $C^s(K)$ be $C(K)$ if $s = 0$ and $C^1(K)$ if $s = 1$. Moreover, let $Z_s(f)$ be $Z(f)$ if $s = 0$ and $Z_1(f)$ if $s = 1$.

We first recall the unicity results of Section 1 for our special problem.

THEOREM 2.1. *Let Problem II be given. Let* $\mathbf{p} \in \mathbf{P}$ *and suppose that* $s = 0$ *or* $s = 1$. *If* $s = 1$ *we assume that* K *is a compact, convex set with a piecewise smooth boundary. Let* U *be in* $C^s(K)$. *Then the following conditions are equivalent*:

(a) *Problem II has a unique solution for all* $f \in C^s(K)$.

(b) *For all* $\mathbf{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ *there is a* $\mathbf{b} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ *such that*

$$\sum_{i=1}^n b_i u_i(x) \leqslant 0 \qquad \text{for all} \quad x \in Z_s\left(\sum_{i=1}^n a_i u_i\right),$$

$$p(\mathbf{b}) < 0.$$

(c) *There do not exist a vector* $\mathbf{a} = (a_1, ..., a_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, *points* $\{x_j\}_{j=1}^r$ *in* $Z_s(\sum_{i=1}^n a_i u_i)$, *and positive numbers* $\{\lambda_j\}_{j=1}^r$, $r \leqslant n$, *satisfying*

$$\sum_{j=1}^r \lambda_j u_i(x_j) = -p_i, \qquad i = 1, ..., n.$$

Remark. If $s = 0$, i.e., only continuous functions are considered, this theorem is due to Nürnberger [6, 7].

These conditions are often not easy to check. We obtain better results if we use the relationship between semi-infinite optimization and one-sided L^1-approximation and apply well-known results of this theory.

The next result is due to Pinkus [11]. We prove it here for completeness.

PROPOSITION 2.2. *Let Problem II be given. Then* $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}$ *if and only if there is a* $w \in C(K)$, $w(x) > 0$, *for all* $x \in K$, *satisfying*

$$\int_K u_i(x)\, w(x)\, dx = -p_i, \qquad i = 1, ..., n.$$

Proof. Assume that there does not exist such a $w \in C(K)$. Then

$$-\mathbf{p} \notin A = \left\{\left(\int_K u_1 w, ..., \int_K u_n w\right) : u_i \in C(K), \quad i = 1, ..., n, \ w > 0\right\}.$$

A is a convex cone and hence there exists a $\mathbf{b} = (b_1, ..., b_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ satisfying

$$\sum_{i=1}^{n} b_i \left(\int_K u_i w \right) \leqslant 0 \leqslant - \sum_{i=1}^{n} b_i p_i$$

for all $w > 0$. This implies that $\sum_{i=1}^{n} b_i u_i(x) \leqslant 0$ for all $x \in K$. Hence $\mathbf{p} \notin \mathbf{P}$.

For the converse let $w \in C(K)$, $w > 0$, be such that $\int_K u_i w = -p_i$, $i = 1, ..., n$. Assume that there exists a $\mathbf{b} \neq 0$ satisfying

$$\sum_{i=1}^{n} b_i u_i(x) \leqslant 0, \qquad x \in K$$

$$\sum_{i=1}^{n} b_i p_i \leqslant 0.$$

Then $-\sum_{i=1}^{n} b_i p_i = \int_K (\sum_{i=1}^{n} b_i u_i) w \geqslant 0$. On the other hand, it follows from $\sum_{i=1}^{n} b_i u_i(x) \leqslant 0$ for all $x \in K$ and $\sum_{i=1}^{n} b_i u_i \neq 0$ that $\int_K (\sum_{i=1}^{n} b_i u_i) w < 0$. This contradiction completes the proof. ∎

We now obtain the following relationship between linear semi-infinite optimization and one-sided L_1-approximation.

PROPOSITION 2.3. *Let Problem II be given and let* $\mathbf{p}$ *be in* $\mathbf{P}$. *Then there exists a function* $w \in C(K)$, $w > 0$, *such that Problem II is equivalent to the following problem: Determine* $\mathbf{a}^* = (a_1^*, ..., a_n^*) \in \mathbb{R}^n$ *satisfying* $\sum_{i=1}^{n} a_i^* u_i \leqslant f$ *such that*

$$\int_K \left(f - \sum_{i=1}^{n} a_i u_i \right) w \geqslant \int_K \left(f - \sum_{i=1}^{n} a_i^* u_i \right) w$$

for all $\mathbf{a} \in \mathbb{R}^n$ *satisfying* $\sum a_i u_i \leqslant f$.

Proof. Since $\mathbf{p} \in \mathbf{P}$ it follows from Proposition 2.2 that there exists a $w \in C(K)$, $w > 0$, such that $\int_K u_i w = -p_i$, $i = 1, ..., n$. Hence $\mathbf{a}^*$ satisfies $\sum_{i=1}^{n} a_i p_i \geqslant \sum_{i=1}^{n} a_i^* p_i$ for all $\mathbf{a}$ satisfying $\sum a_i u_i \leqslant f$. This problem is equivalent to the above one-sided L_1-approximation problem. ∎

It is obvious that every one-sided L^1-approximation problem can be considered as a semi-infinite optimization problem.

We now apply this relationship.

THEOREM 2.4. *Let Problem II be given,* $\mathbf{p} \in \mathbf{P}$, *and assume that* $\mathrm{int}(K)$ *is connected. Then there is a function* $f \in C(K)$ *such that Problem II has at least two solutions.*

Proof. Since $\mathbf{p} \in \mathbf{P}$ it follows from Proposition 2.3 that Problem II can be considered as a one-sided L_1-approximation problem. Then the theorem follows from Theorem 5.13 in [8, p. 109]. ∎

Remark. Further theorems of this type can be obtained using results in [8, p. 110]. Theorem 2.4 is a negative result. Therefore we study what can be said about the dimension of the set of solutions.

In the following the set of solutions of Problem II for fixed $\mathbf{p}$ and f (and of course fixed U) is denoted by $E(\mathbf{p}, f)$.

DEFINITION 2.5. Let Problem II be given and let $\mathbf{p} \in \mathbf{P}$. Then the set of solutions $E(\mathbf{p}, f)$ is said to have *dimension* k $(0 \leqslant k \leqslant n-1)$ if there exist $k+1$ vectors $\mathbf{a}^0, ..., \mathbf{a}^k$ in $E(\mathbf{p}, f)$ such that $\mathbf{a}^0 - \mathbf{a}^1, ..., \mathbf{a}^0 - \mathbf{a}^k$ are linearly independent and k is maximal with respect to this property.

Let the vectors $\mathbf{a}^0, ..., \mathbf{a}^k$ in $\mathbb{R}^n$, $\mathbf{a}^j = (a_1^j, ..., a_n^j)$, be given and let $U = \operatorname{span}\{u_1, ..., u_n\}$ be in $C(K)$. Then we define

$$Z(\mathbf{a}^0, ..., \mathbf{a}^k, U) = \bigcap_{j=0}^{k} Z\left(\sum_{i=1}^{n} a_i^j u_i\right).$$

We shall need the following condition.

DEFINITION 2.6. Let $U = \operatorname{span}\{u_1, ..., u_n\}$, $n \geqslant 2$, be a subspace of $C(K)$ which contains a strictly positive function. U is said to satisfy *Property* D^k, $0 \leqslant k \leqslant n-1$, if for linearly independent vectors $\mathbf{a}^0, ..., \mathbf{a}^k$ there exists a $v \in U \setminus \{0\}$ satisfying

$$Z(\mathbf{a}^0, ..., \mathbf{a}^k, U) \subseteq Z(v),$$
$$v(x) \geqslant 0, \qquad x \in K.$$

Remark. In the theory of best L_1-approximation, subspaces satisfying Property A, B, or C (see [8, 10, 14, 15]) have been introduced. These are conditions ensuring uniqueness for certain L_1-approximation problems. The definition of subspaces satisfying Property D^k is in the same spirit. Later we shall also consider subspaces satisfying Property B (or B^k).

Now we are in position to give the main results of this section. We first consider the case of continuous functions.

THEOREM 2.7. *Let Problem II be given. Then* $\dim E(\mathbf{p}, f) \leqslant k$ *for all* $\mathbf{p} \in \mathbf{P}$ *and all* $f \in C(K)$ *if and only if* U *satisfies Property* D^k, $0 \leqslant k \leqslant n-1$.

Proof. Let us first assume that there exist $k+1$ vectors $\mathbf{a}^0, \dots, \mathbf{a}^k$, $\mathbf{a}^j = (a_1^j, \dots, a_n^j)$ which are linearly independent and $Z(\mathbf{a}^0, \dots, \mathbf{a}^k, U) \subseteq Z(v)$, $v(x) \geqslant 0$ for all $x \in K$, $v \in U$, implies that $v \equiv 0$.

We set

$$H = \left\{ \mathbf{a} = (a_1, \dots, a_n) : \sum_{i=1}^{n} a_i u_i(x) \leqslant 0 \text{ for all } x \in Z(\mathbf{a}^0, \dots, \mathbf{a}^k) \right\}.$$

Then it can be shown as in [8, p. 120] that there is a $w \in C(K)$, $w > 0$, satisfying

$$\int_K \left(\sum_{i=1}^{n} a_i u_i(x) \right) w(x)\, dx \leqslant 0$$

for all $\mathbf{a} = (a_1, \dots, a_n)$ in H. Set $p_i = -\int_K u_i w$; then it follows from Lemma 1.4 that there exist points $x_1, \dots, x_r$ in $Z(\mathbf{a}^0, \dots, \mathbf{a}^k, U)$, $1 \leqslant r \leqslant n$, and $\lambda_1 > 0, \dots, \lambda_r > 0$ such that

$$\int_K u_i(x)\, w(x)\, dx = \sum_{j=1}^{r} \lambda_j u_i(x_j), \qquad i = 1, \dots, n.$$

We now define

$$v_j = \sum_{i=1}^{n} a_i^j u_i, \qquad j = 0, \dots, k.$$

Then $F = \sum_{j=0}^{k} |v_j|$ satisfies $F \in C(K)$, $F \geqslant 0$, and

$$F(x_j) = 0, \qquad j = 1, \dots, r,$$
$$F(x) \geqslant |v_j(x)|, \qquad j = 0, \dots, k.$$

Suppose that $v \in U$ satisfies $v(x) \leqslant F(x)$, $x \in K$. Then $v(x_i) \leqslant 0$, $i = 1, \dots, r$, and therefore

$$\int_K v(x)\, w(x)\, dx = \sum_{j=1}^{r} \lambda_j v(x_j) \leqslant 0.$$

Since $v_j(x) = 0$ for all $x \in Z(\mathbf{a}^0, \dots, \mathbf{a}^k, U)$ we have that

$$\int_K v_j(x)\, w(x)\, dx = 0, \qquad j = 0, \dots, k.$$

If in Problem II we set $f = F$ and $\mathbf{p} = (p_1, \dots, p_n)$, where $p_i = -\int_K u_i w$, $i = 1, \dots, n$, then $\mathbf{0}$, $\mathbf{a}^0, \dots, \mathbf{a}^k$ are solutions. This contradicts the fact that $\dim E(\mathbf{p}, f) \leqslant k$. To prove the converse we assume that there exist $\mathbf{p} \in \mathbf{P}$

and $f \in C(K)$ such that $\dim E(\mathbf{p}, f) \geqslant k+1$. Hence there exist vectors $\mathbf{a}^0, ..., \mathbf{a}^{k+1}$ in $E(\mathbf{p}, f)$ such that $\mathbf{b}^i = \mathbf{a}^i - \mathbf{a}^0$, $i = 1, ..., k+1$, are linearly independent. Let us define $v_j = \sum_{i=1}^n a_i^j u_i$, $\mathbf{a}^j = (a_1^j, ..., a_n^j)$, for $j = 0, ..., k+1$.

Let $F = f - (1/(k+2)) \sum_{j=0}^{k+1} v_j = (1/(k+2)) \sum_{j=0}^{k+1} (f - v_j)$. Since $f - v_j \geqslant 0$ it follows that $F(\tilde{x}) = 0$ for some $\tilde{x} \in K$ implies that $f(\tilde{x}) = v_0(\tilde{x}) = \cdots = v_{k+1}(\tilde{x})$. Hence $Z(F) \subset Z(\mathbf{b}^1, ..., \mathbf{b}^{k+1}, U)$. Moreover, $\mathbf{0} \in E(\mathbf{p}, F)$.

Since U is a D^k-space there exists a $v = \sum_{i=1}^n c_i u_i \in U \backslash \{0\}$ satisfying

$$Z(\mathbf{b}^1, ..., \mathbf{b}^{k+1}, U) \subset Z(v),$$

$$v(x) \geqslant 0, \qquad x \in K.$$

It follows from Proposition 2.2 that there is a $w \in C(K)$, $w > 0$, on K such that $\int_K u_i(x)\, w(x)\, dx = -p_i$, $i = 1, ..., n$. Therefore

$$p(\mathbf{c}) = \sum_{i=1}^n c_i p_i = -\sum_{i=1}^n \left(c_i \int_K u_i(x)\, w(x)\, dx\right) = -\int_K v(x)\, w(x)\, dx < 0.$$

Since U contains a strictly positive function we can construct a function $u = \sum_{i=1}^n d_i u_i$ such that $u(x) \leqslant F(x)$, $x \in K$, and

$$\sum_{i=1}^n d_i p_i = -\sum_{i=1}^n d_i \int_K u_i(x)\, w(x)\, dx = -\int_K u(x)\, w(x)\, dx < 0.$$

This is a contradiction to the fact that $\mathbf{0} \in E(\mathbf{p}, F)$. ∎

We obtain another property of these subspaces.

THEOREM 2.8. *Suppose that the subspace $U = \text{span}\{u_1, ..., u_n\}$ in $C(K)$ satisfies Property D^k, $0 \leqslant k \leqslant n-1$. Then there does not exist an integration rule $Q(f) = \sum_{i=1}^m a_i f(x_i)$, $a_i \in \mathbb{R}$, $x_i \in K$ for $i = 1, ..., m$, which satisfies $Q(u) = \int_K u(x)\, w(x)\, dx$ for all $u \in U$ and $m \leqslant n-k-1$.*

Proof. Let there be given any $n-k-1$ distinct points $t_1, ..., t_{n-k-1}$ in K. Then there are $k+1$ linearly independent functions $v_j = \sum_{i=1}^n a_i^j u_i$, $j = 1, ..., k+1$, such that $v_j(t_i) = 0$, $i = 1, ..., n-k-1$ and $j = 1, ..., k+1$. Hence $\{t_i\}_1^{n-k-1} \subset Z(\mathbf{a}^1, ..., \mathbf{a}^{k+1}, U)$, $\mathbf{a}^j = (a_1^j, ..., a_n^j)$. Since U is a D^k-space there exist a $v \in U \backslash \{0\}$ such that $v(t_i) = 0$, $i = 1, ..., n-k-1$, and $v(x) \geqslant 0$, $x \in K$. Then it follows for every integration rule with the knots $\{t_i\}_{i=1}^{n-k-1}$ that $Q(v) = \sum_{i=1}^{n-k-1} a_i v(t_i) = 0$. On the other hand, $\int_K v(x)\, w(x)\, dx > 0$. Hence the rule Q is not exact for all $u \in U$. ∎

We now consider a simple example.

PROPOSITION 2.9. *Let U be an n-dimensional subspace of $C[a, b]$, $n \geqslant 2$. Suppose that U is a Haar space (i.e., every $u \in U \backslash \{0\}$ has at most $n-1$ distinct zeros). Then U satisfies Property $D^{[n/2]}$ but not Property $D^{[n/2]-1}$.*

Proof. Assume that U satisfies Property $D^{[n/2]-1}$. Then it follows from Theorem 2.8 that there is no quadrature formula which is exact for all $u \in U$ and has at most $n-([n/2]-1)-1=n-[n/2]$ knots. This is a contradiction since Gaussian quadrature formulae have exactly $n-[n/2]$ knots.

We now prove that U satisfies Property $D^{[n/2]}$. Assume that $v_1, ..., v_{[n/2]+1}$ are linearly independent functions in U. Since U is a Haar space the set $\{x : v_i(x)=0, i=1, ..., [n/2]+1\}$ contains at most $h=n-[n/2]-1$ points $\{t_i\}_{i=1}^{d}$ (i.e., $d \leqslant h$). It follows from well-known results on Haar spaces (see [3, p. 28]) that there is an $h \in U \setminus \{0\}$, $h \geqslant 0$, satisfying $h(t_i)=0$, $i=1, ..., d$. Hence U satisfies Property $D^{[n/2]}$. ∎

Theorem 2.4 and Proposition 2.9 are negative results concerning uniqueness of the solutions. But the situation completely differs if we consider differentiable functions. In addition, we now consider problems which have a unique solution not only for all $f \in C^1(K)$ but also for all $\mathbf{p} \in \mathbf{P}$. To this end, we define the following condition.

DEFINITION 2.10. Let K be a compact convex subset of $\mathbb{R}^d$ with piecewise smooth boundary. Suppose that U is an n-dimensional subspace of $C^1(K)$ which contains a strictly positive function. Then U is said to satisfy *Property B* if to each $u \in U$, $u \neq 0$, there exists a $v \in U$, $v \neq 0$, for which

$$Z_1(u) \subset Z_1(v),$$
$$v \geqslant 0.$$

We obtain the following characterization theorem.

THEOREM 2.11. *Let Problem II be given, $n \geqslant 2$. Suppose that K and U satisfy the conditions in Definition* 2.10. *Then there exists a unique solution of Problem II for all $f \in C^1(K)$ and all $\mathbf{p} \in \mathbf{P}$ if and only if U satisfies Property B.*

Proof. It follows from Proposition 2.3 that Problem II is equivalent to a one-sided L_1-approximation problem. Then the theorem follows from Theorems 5.19 and 5.22 in [8]. ∎

Subspaces satisfying Property B have been considered in one-sided L^1-approximation in detail. We give some examples.

EXAMPLES. (1) Let U be an n-dimensional Haar space in $C^1[a, b]$ and for every $u \in U$, $u \neq 0$, we have $I(Z_1(u)) \leqslant n-1$, where $I(A)$ counts the number of points in A under the convention that points in $A \cap (a, b)$ are counted twice and points in $A \cap \{a, b\}$ once. Then U satisfies Property B.

(2) Polynomial splines with simple fixed knots. Let an interval $[a, b]$ and the knots $a = x_0 < x_1 < \cdots < x_r < x_{r+1} = b$ be given. Consider the space $S_{m-1,r}[a, b]$, $m \geq 3$, of functions s which on each of the intervals $[x_{i-1}, x_i]$, $i = 1, ..., r+1$, are polynomials of degree at most $m-1$ and such that $s \in C^{(m-2)}[a, b]$. This means that we have differentiable functions. Then $S_{m-1,r}[a, b]$ satisfies Property B.

Proofs and further examples can be found in [8, p. 121]. See also [12]. It should be mentioned that only a few examples of subspaces satisfying Property B are known if $K \subset \mathbb{R}^d$, $d > 1$.

We can also give a characterization of the dimension of the solution set for differentiable functions.

Let K be a compact convex subset of $\mathbb{R}^d$ with piecewise smooth boundary. For any set of vectors $\mathbf{a}^0, ..., \mathbf{a}^k$ in $\mathbb{R}^n$ and given $U = \text{span}\{u_1, ..., u_n\}$ in $C^1(K)$ we define

$$Z_1(\mathbf{a}^0, ..., \mathbf{a}^k, U) = \bigcap_{j=0}^{k} Z_1\left(\sum_{i=1}^{n} a_i^j u_i\right).$$

DEFINITION 2.12. Let $U = \text{span}\{u_1, ..., u_n\}$ be a subspace of $C^1(K)$ which contains a strictly positive function. U is said to satisfy *Property* B^k, $0 \leq k \leq n-1$, if for any linearly independent vectors $\mathbf{a}^0, ..., \mathbf{a}^k$ there exists a $v \in U \setminus \{0\}$ satisfying

$$Z_1(\mathbf{a}^0, ..., \mathbf{a}^k, U) \subseteq Z_1(v),$$
$$v(x) \geq 0, \qquad x \in K.$$

THEOREM 2.13. *Let Problem II be given,* $\mathbf{p} \in \mathbf{P}$, *and* K *be a compact convex subset of* $\mathbb{R}^d$ *with piecewise smooth boundary. Then* $\dim E(\mathbf{p}, f) \leq k$ *for all* $\mathbf{p} \in \mathbf{P}$ *and* $f \subset C^1(K)$ *if and only if* U *satisfies Property* B^k $(0 \leq k \leq n-1)$.

Proof. The theorem follows from Proposition 2.3 and Theorem 1.4 in [13]. ∎

In [13] there are also given examples of subspaces which satisfy Property B^k including classes of Haar spaces defined on disjoint intervals, spaces of lacunary polynomials, and certain spaces of bivariate polynomials.

Remarks 2.14. The results in [5] can also be used—together with Proposition 2.3—to obtain theorems on strong uniqueness of order γ $(\gamma \geq 1)$ for the solutions in semi-infinite optimization.

Remark 2.15. The above given theory has been shown for the case $m = 1$. But we were not able to extend the results to the case $m > 1$. The following example of linear best Chebyshev approximation with constraints shows that the case $m > 1$ is very complicated.

PROBLEM III. *Let K be a compact convex set in $\mathbb{R}^d$ with piecewise smooth boundary. Assume that $f \in C(K)$ and U is an n-dimensional subspace of $C(K)$, $n \geqslant 2$. Let $\{v_1^j, ..., v_n^j\}$, $j = 1, ..., r$, $r \geqslant 0$, and $\{h_1, ..., h_r\}$ be sets of functions in $C^s(K)$ where $s = 0$ or $s = 1$. Suppose that*

$$Q = Q(h_1, ..., h_r) = \left\{ \mathbf{b} \in \mathbb{R}^n : \sum_{i=1}^{n} b_i v_i^j(x) \leqslant h_j(x), x \in K, j = 1, ..., r \right\}$$

and $\operatorname{int}(Q)$ *is nonempty. Determine a best approximation $u^* = \sum_{i=1}^n b_i^* u_i$ in U, $\mathbf{b}^* = (b_1^*, ..., b_n^*)$, in Q satisfying*

$$\| f - u^* \|_\infty \leqslant \| f - u \|_\infty$$

for all $u = \sum_{i=1}^n b_i u_i$, $\mathbf{b} = (b_i)_1^n$ in Q, and $\| f \|_\infty = \max_{x \in K} |f(x)|$.

This is equivalent to the following optimization problem.

PROBLEM III'. *Let the assumptions of Problem III be given. Minimize $p(\mathbf{a}) = c$, $\mathbf{a} = (b_1, ..., b_n, c)$, subject to the constraints*

$$\text{(a)} \quad \sigma \sum_{i=1}^{n} b_i u_i(x) - c \leqslant \sigma f(x), \qquad \sigma \in \{-1, 1\}$$

$$\text{(b)} \quad \sum_{i=1}^{n} b_i v_i^j(x) \leqslant h_j(x), \qquad j = 1, ..., r$$

for all $x \in K$.

This is a semi-infinite optimization problem which has the form of Problem I. We have $\mathbf{p} = (0, ..., 0, 1)$ and $\mathbf{a} = (b_1, ..., b_n, c)$ in $\mathbb{R}^{n+1}$. Since $\operatorname{int}(Q) \neq \emptyset$ there exists an $\tilde{\mathbf{a}} = (\tilde{b}_1, ..., \tilde{b}_n, \tilde{c})$ such that (a)–(b) in Problem III' satisfy the Slater condition. It follows immediately from Problem III that there exists a best approximation and that the solution set is bounded. Hence this is also true for Problem III' and therefore $\mathbf{p} \in \mathbf{P}$.

Remark 2.16. Characterizations of best approximations and a Haar type theory of Problem III have already been given in [4]. We can also obtain these results if we apply Theorem 1.3 and Corollary 1.6 to Problem III'.

If $r = 0$ in Problem III then we have no constraints. The well-known Haar theory characterizes unicity. In particular, we have uniqueness for $U \in C(K)$ and all $f \in C(K)$ if U satisfies the Haar condition. In contrary to the case of Problem I with $m = 1$ we also have positive results in this problem if we use continuous functions.

If $r \geqslant 1$ in Problem III then requiring uniqueness for continuous functions $\{v_i^j\}$ and $\{h_j\}$ is too restrictive. On the other hand it is shown in [4] that the variety of subspaces providing positive results is much wider if $\{v_i^j\}$ and $\{h_j\}$ are smooth functions.

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